Simpson's Theory and Superrigidity of Complex Hyperbolic Lattices

ALEXANDER REZNIKOV

Abstract We attack a conjecture of J. Rogawski: any cocompact lattice in SU(2,1) for which the ball quotient $X = B^2/\Gamma$ satisfies $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X,\mathbb{Q}) \approx \mathbb{Q}$ is arithmetic. We prove the Archimedian suprerigidity for representation of Γ is $SL(3,\mathbb{C})$.

Théorie de Simpson et superrigidité des réseaux hyperboliques complexes

Résumé Soit $\Gamma \subset SU(2,1)$ un reseau cocompact et soit $X = B^2/\Gamma$. Nous preuvons: si $b_1(X) = 0$ et $H^{1,1}(X) \cap H^2(X,\mathbb{Q}) \approx \mathbb{Q}$ allors tous les representations ρ de Γ dans $SL(3,\mathbb{C})$ sont conjugué à le représentation naturelle ou la fermeture de Zariski de l'image $p(\Gamma)$ est compacte.

Version française abrégéé - Le théorème classique de Margulis dit que tous les réseaux dans les groupes de Lie semi-simples sont superrigides. Ceci a eté generalisé par Corlette [C] à la superrigidite des réseaux quaternioniques et de Cayley. D'autre part, Johnson et Millson ont montré qu'il existait des deformations des réseaux cocompact dans SO(n,1) si on regarde SO(n,1) comme plongé dans SO(n+1,1).

C'est une question d'un intérêt fondamental de savoir si les réseaux hyperboliques complexes sont superrigides.

Dans cet article, nons considerons la question suivante de J. Rogawski.

<u>Hypothése</u> Soit $X = B^2/\Gamma, \Gamma \subset SU(2,1)$ une surface hyperbolique complexe compacte. Supposons $b_1(X) = 0$ et $H^{1,1}(X) \cap H^2(X,\mathbb{Q}) = \mathbb{Q}$. Allors Γ est arithmetique et provient d'une algébre avec division $E|\mathbb{Q}$ de rang 3 avec une involution.

Observons que pour tous les réseux provenant d'algébres avec division, on a effectivement $b_1(X) = 0$ et $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \mathbb{Q}$ [Rog].

Soit ℓ un fibré linéaire tautologiue sur X [Re]. La condition $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \mathbb{Q}$ dit que $[\ell] = k \cdot$ générateur dans $Pic(X)/tors \approx \mathbb{Z}$.

Le résultat principlal de cet article prouve la superrigidité des representations de Γ dans $SL(3,\mathbb{C})$ dans le cas k=1.

Theoremé principal Soit $X = B^2/\Gamma$ et supposons que $b_1(X) = 0$ et $H^{1,1}(X) \cap H^2(X,\mathbb{Q}) = \mathbb{Q}$. Soit $[\ell]$ un générateur de $Pic(X)/tors \approx \mathbb{Z}$. Si ρ est une repréentation de Γ dans $SL(3,\mathbb{C})$ alors soit ρ est conjugué à la représentation naturelle de Γ , soit la fermeture de Zariski de l'image $\rho(\Gamma)$ est compacte.

Je vaudrais remercier Ron Livné, Jon Rogawski, et Carlos Simpson pour beacoup de discussiones intéresantes. Je vaudrais aussi remercier Marina Ville et Lucy Katz pour son aide essentielle à la préparation de cet article.

Simpson's Theory and Superrigidity of Complex Hyperbolic Lattices

Alexander Reznikov

0 Main Theorem The classical theorem of Margulis establishes the superrigidity of lattices in semisimple Lie groups of rank ≥ 2 . The work of Corlette [C] extended this to (Archimedian) superrigidity of uniform quaternionic and Cayley lattices. On the other hand, by Johnson and Millson [JM] some uniform lattices in SO(n,1) admit deformations as mapped to SO(n+1,1).

It is therefore of fundamental interest to study to what extent the complex hyperbolic lattices are superrigid. Since there are nontrivial holomorphic maps between different ball quotients [DM] one should confine oneself's look to lattices (or manifolds) "minimal" in some sense.

The present note addresses the following conjecture of Jon Rogawski.

<u>Conjecture</u>. Let $X = B^2/\Gamma, \Gamma \subset SU(2,1)$ be a compact ball quotient. Suppose $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X,\mathbb{Q}) \approx \mathbb{Q}$. Then Γ is arithmetic and comes from a division algebra $E|\mathbb{Q}$ of rank 3 with an involution.

Observe that for all lattices coming from division algebras, indeed $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \approx \mathbb{Q}$ [Rog].

Let ℓ be the tautological line bundle over X [Re]. Since $Pic(X)/tors \approx \mathbb{Z}$, we have $[\ell] = k \cdot \text{generator for some } k \in \mathbb{Z}$.

The main result of the paper establishes the superrigidity of representations of Γ in $SL(3,\mathbb{C})$ for Γ yielding k=1 as follows.

<u>Main Theorem</u>. Let $X = B^2/\Gamma$ and suppose $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \approx Q$. If $[\ell]$ generates $Pic(X)/tors \approx \mathbb{Z}$, then any representation of $\Gamma = \pi_1(X)$ in $SL(3, \mathbb{C})$ is either conjugate to the natural representation up to the twist by a character, or has a compact Zariski closure.

One hopes, that, applying methods of [GS] one is able to prove the p-adic superrigidity and to settle Rogawski's conjecture.

I wish to thank Ron Livne, Jon Rogawski and Carlos Simpson for stimulating discussions.

1. Computations of Higgs bundles. We admit a knowledge of Simpson's theory [S1]. Let X be as above and let $\rho_0: \Gamma \to PSU(n,1)$ be the natural representation. Then the corresponding Higgs bundle is as follows [Re]. Take $E = TX \otimes \ell \oplus \ell$ as a holomorphis bundle and define $\theta \in H^0(\Omega^1 \otimes End(E))$ by $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

In view of the Simpson's theory, for proving the Main Theorem one needs to show that any complex variation of Hodge structure [S1] of type (2,1) over X is as above. Indeed, any representation is deformable to one, corresponding to a variation of Hodge structure [S2], and the natural representation is rigid [W].

So let $F = (\xi \oplus \eta, \theta)$ be a variation of complex Hodge structure, rank $\xi = 2$, rank $\eta = 1, \theta \in H^0(\Omega^1(X) \oplus \operatorname{Hom}(\eta, \xi)) \approx H^0(\operatorname{Hom}(TX \otimes \xi, \eta))$.

1.2. <u>Lemma</u>. Let λ, μ be rank two bundles over X and let $f \in H^0(Hom(\lambda, \mu)), f \neq 0$. Then either rank $f \leq 1$ everywhere or

$$(c_1(\mu), [\omega]) \ge (c_1(\lambda), [\omega])$$

with the equality iff $\lambda \approx \mu$, and rank f = 2 everywhere. Here $[\omega]$ is the Kähler class.

PROOF: Consider $\wedge^2 f: \wedge^2 \lambda \to \wedge^2 \mu$. If $\wedge^2 f \neq 0$, then $\wedge^2 \mu \oplus (\wedge^2 \lambda)^{-1}$ has a nontrivial holomorphic section, whose zero locus is an effective divisor, so $(c_1(\wedge^2 \mu \otimes (\wedge^2(\lambda))^{-1}), [\omega]) \geq 0$ and the equality implies $\wedge^2 f$ is an isomorphism.

2. Proof of the Main Theorem:

Let $F = (\xi \oplus \eta, \theta)$ be as above.

<u>Case 1</u> Rank $\theta = 2$ somewhere.

Applying the lemma, we get

$$(c_1(TX \otimes \xi), [\omega]) \leq (c_1(\eta), [\omega])$$

Now, $[\omega] \sim [\ell]$ since X is hyperbolic, and $c_1(TX) = -3[\ell]$ in $H^2(X, \mathbb{R})$, so

$$(c_1(\eta) - 2c_1(\xi), [\ell]) \le 3[\ell]^2.$$

On the other hand, since $\xi \oplus \eta$ is a deformation of the flat bundle, $c_1(\xi \oplus \eta) = 0$, i.e. $c_1(\xi) = -c_1(\eta)$, so

(*)
$$(c_1(\xi), [\ell]) \leq [\ell]^2.$$

Since F is θ -stable [S1], $(c_1(\eta), [\ell]) < 0$, so $(c_1(\eta), [\ell]) > 0$. This leaves the only possibility $(c_1(\xi), [\ell]) = [\ell]^2$, because $[\ell]$ generates Pic(X)/tors. So $\xi = \ell \otimes \alpha$, where α is a linear unitary flat bundle, corresponding to $Pic(X)/tors \approx H_1^{tors}(X, \mathbb{Z})$. (recall that $b_1(X) = 0$). Moreover, since (*) becomes an equality, we get by lemma above $\eta \approx TX \otimes \xi$ and θ takes the form $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $F \approx (TX \otimes \ell \oplus \ell) \otimes \alpha$ and the proof is complete in this case.

Case 2 Rank $\theta \leq 1$ everywhere on X. There exists a collection of points (p_1, \dots, p_k) such that Ker θ extends to a rank one subbundle of $TX \otimes \xi$, say $\alpha \otimes \eta$. Since $H^2(X - \{p_1, \dots, p_k\}) \approx H^2(X)$, $c_1(\alpha)$ is well-defined in $H^2(X, \mathbb{Z})$. Moreover, by the removing of singularities in codimension two we have $H^i(X, \mathcal{O}) \approx H^i(X - \{p_1 \cdots p_k\}, \mathcal{O})$, so from the exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$ and the five-lemma we deduce that $H^1(X, \mathcal{O}^*) \approx H^1(X - \{p_1, \dots, p_k\}, \mathcal{O}^*)$, so $c_1(\alpha)$ is in the image of Pic(X) in $H^2(X, \mathbb{Z})$. Let C be an irreducible curve of sufficiently high degree, which does not meet p_1, \dots, p_k . Since $TX \otimes \ell \oplus \ell$ remains θ -stable on C [S1] we get $(c_1(\alpha \otimes \ell)|_{C}, [C]) < 0$. In view of $H^{1,1}(X) \cap H^2(X_1\mathbb{Q}) \approx \mathbb{Q}$ we can rewrite this as $(c_1(\alpha), [\ell]) < -[\ell]^2$. Since $[\ell]$ generates Pic(X)/tors, this actually means $(c_1(\alpha), [\ell]) \leq -2[\ell]^2$. Now, $c_1(TX) = -3[\ell]$, so $(c_1(TX/\alpha), [\ell]) \geq -[\ell]^2$. On C we have an isomorphism

$$\theta|_C: TX|\alpha \otimes \xi \to Im\theta \subset \eta|_C.$$

Hence $(c_1(\operatorname{Im}\theta), [C]) = (c_1(TX/\alpha) + c_1(\xi), [C]) \ge (c_1(\xi) - [\ell], [C])$. Since, again, $[\ell]$ generates Pic(X)/tors and $(c_1(\xi), [\ell] > 0$ we get $(c_1(\xi) - [\ell], [C]) \ge 0$. This contradicts the θ -stability of $\xi \oplus \eta|_C$, because $\operatorname{Im}\theta|_C$ is θ -invariant. The proof is complete.

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Institute of Mathematics

Hebrew University

Givat Ram 91904

ISRAEL

email: simplex@sunset.huji.ac.il